International Conference
20th EURO Mini Conference
"Continuous Optimization and Knowledge-Based Technologies"
(EurOPT-2008)
May 20–23, 2008, Neringa, LITHUANIA

ISBN 978-9955-28-283-9
L. Sakalauskas, G.W. Weber and
E. K. Zavadskas (Eds.): EUROPT-2008
Selected papers. Vilnius, 2008, pp. 229–234


# ON BI-PARAMETRIC PROGRAMMING IN QUADRATIC OPTIMIZATION

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**Abstract:** In this paper we consider the Convex Quadratic Optimization problem with simultaneous perturbation in the right-hand-side of the constraints and the linear term of the objective function with different parameters. The regions with invariant optimal partitions are investigated as well as the behaviour of the optimal value function on the regions. We show that identifying these regions can be done in polynomial time in the output size. An algorithm for identifying all invariancy regions is presented. Interior Point Methods are employed for solving sub-problems. Some implementation details, as well as a numerical example are discussed.

**Keywords:** bi-parametric optimization, convex quadratic optimization, interior-point methods, optimal partition, invariancy region.

## 1. Introduction

Parametric programming is a technique that allows determining how the optimal objective function value and an optimal solution varies with the change in one or more coefficients (parameters) of the objective function or right-hand side values of the problem constraints. Consequently, it allows obtaining optimal solutions as a function of parameters appearing in the optimization problem. Parametric optimization is a generalization of sensitivity analysis (Berkelaar *et al.*, 1997). In addition, multi-objective and stochastic optimization models can be formulated as parametric programming problems (Guddat *et al.*, 1985).

In this paper we are concerned with the parametric analysis of Convex Quadratic Optimization (CQO) problems where the coefficient vector of the linear term of the objective function and the right-hand side vector of the constraints are varied independently. The Bi-Parametric CQO problem is given as

$$\min\left\{(c+\lambda\Delta c)^T x + 1/2x^T Qx : Ax = b+\varepsilon\Delta b, x\geq 0\right\}, \tag{1}$$

where $A\in\Re^{m\times n}$, $Q\in\Re^{n\times n}$ is a symmetric semi-definite matrix, $b\in\Re^m$ and $c\in\Re^n$ are fixed data, $\varepsilon$ and $\lambda$ are two real parameters, $\Delta b\in\Re^m$, $\Delta c\in\Re^n$ are the perturbation directions and $x\in\Re^n$ is an unknown vector. The dual of problem (1) is

$$\max\left\{(b+\varepsilon\Delta b)^T y - 1/2x^T Qx : A^T y + s - Qx = c+\lambda\Delta c, x\geq 0, s\geq 0\right\}, \tag{2}$$

where $y\in\Re^m$ and $s\in\Re^n$ are unknowns. Any $x(\varepsilon)\geq 0$ satisfying $Ax=b+\varepsilon\Delta b$ is called a *primal feasible* solution of problem (1). Further, a vector $(x(\lambda),y(\lambda),s(\lambda))$ with $x(\lambda),s(\lambda)\geq 0$ is called a *dual feasible* solution of (2) if it satisfies $A^Ty+s-Qx=c+\lambda\Delta c$. The equation $x(\varepsilon)^Ts(\lambda)=0$ is known as the *complementarity condition*. Let $\mathcal{QP}(\Delta b,\Delta c,\varepsilon,\lambda)$ and $\mathcal{QP}^*(\Delta b,\Delta c,\varepsilon,\lambda)$ denote the sets of primal feasible and primal optimal solutions of (1), respectively. Similar notation is used for the sets of feasible and optimal solutions of (2). For the case when $\varepsilon=\lambda=0$, the CQO problem is referred to as the unperturbed problem. Moreover, when either $\Delta b$ or $\Delta c$ is a zero vector, or when $\varepsilon=\lambda$, the perturbed CQO problem is referred to as uni-parametric CQO problem. We consider the general case when $\varepsilon$ and $\lambda$ are not identical and refer to it as the bi-parametric CQO problem. We denote a primal-dual optimal solution of the perturbed CQO problems by $(x^*(\varepsilon,\lambda)$, $y^*(\varepsilon,\lambda)$, $s^*(\varepsilon,\lambda))$.

The *support* set of a nonnegative vector $v$ is defined as $\sigma(v)=\{i : v_i>0\}$. Unlike the case of Linear Optimization (LO), where the index set $\{1,2, \ldots ,n\}$ can be partitioned into two subsets, the index set for CQO needs to be partitioned into three subsets as

$$\mathcal{B}(\varepsilon,\lambda) = \{i : x_i^*(\varepsilon,\lambda)>0 \text{ for an optimal solution } x^*(\varepsilon,\lambda)\},$$

$$\mathcal{N}(\varepsilon,\lambda) = \{i : s_i^*(\varepsilon,\lambda)>0 \text{ for an optimal solution } (x^*(\varepsilon,\lambda),y^*(\varepsilon,\lambda),s^*(\varepsilon,\lambda))\},$$

$$\mathcal{T}(\varepsilon,\lambda) = \{1,2,\ldots,n\} \setminus \mathcal{B}(\varepsilon,\lambda) \cup \mathcal{N}(\varepsilon,\lambda).$$

We denote this partition by $\pi(\varepsilon,\lambda)=(\mathcal{B}(\varepsilon,\lambda), \mathcal{N}(\varepsilon,\lambda), \mathcal{T}(\varepsilon,\lambda))$ and refer to it as the *optimal partition*. The optimal partition $\pi(\varepsilon,\lambda)$ is unique (Berkelaar *et al.*, 1997). A *maximally complementary solution* $(x^*(\varepsilon,\lambda), y^*(\varepsilon,\lambda), s^*(\varepsilon,\lambda))$ is a primal-dual optimal solution of (1) and (2) for which $x_i^*(\varepsilon,\lambda)>0$ if and only if $i \in \mathcal{B}(\varepsilon,\lambda)$,

$$s_i^*(\varepsilon,\lambda)>0 \text{ if and only if } i \in \mathcal{N}(\varepsilon,\lambda).$$

The existence of maximally complementary optimal solutions is a consequence of the convexity of the optimal solution sets $\mathcal{QP}^*(\Delta b,\Delta c,\varepsilon,\lambda)$ and $\mathcal{QD}^*(\Delta b,\Delta c,\varepsilon,\lambda)$. Interior Point Methods (IPMs) are widely used to solve CQO problems in polynomial time (see Berkelaar *et al.*, 1997 and the references therein) and sufficiently accurate solutions obtained by an IPM can be used to produce maximally complementary solutions (Illés *et al.*, 2000). By knowing a maximally complementary optimal solution, one can identify the optimal partition. If for a given optimal partition $\mathcal{T}(\varepsilon,\lambda)=\emptyset$ holds, then any maximally complementary solution is *strictly complementary*.

Let $\phi(\varepsilon,\lambda)$ denote the optimal value function which is defined as:

$$\phi(\varepsilon,\lambda) = (c+\lambda\Delta c)^T x^*(\varepsilon,\lambda) + 1/2 x^*(\varepsilon,\lambda)^T Q x^*(\varepsilon,\lambda). \tag{3}$$

In *optimal partition invariancy* sensitivity analysis we aim to identify the range of parameters where the optimal partition remains invariant. The cases when either $\Delta b$ or $\Delta c$ is zero has been studied in (Berkelaar *et al.*, 1997). The situation when $\varepsilon=\lambda$ has been investigated in (Ghaffari-Hadigheh *et al.*, 2007). In these cases the region of the parameter is an interval of the real line called *invariancy interval*. We refer to the studies mentioned above as *uni-parametric* optimal partition invariancy sensitivity analysis.

Bi-parametric optimal partition sensitivity analysis has been studied in case of LO in (Ghaffari-Hadigheh *et al.*, 2008a, Guddat *et al.*, 1985). Earlier studies on the subject were published in Eastern Germany in German, the interested reader can find their summary in the two publications above. Here, we just point out the most recent results from (Ghaffari-Hadigheh *et al.*, 2008a) in a nutshell. The crucial difference of the bi-parametric LO case from the CQO case is that in the LO case the invariancy regions are open rectangles while in the CQO case those are open convex polygons. In bi-parametric LO the invariancy regions generate a mesh-like area in $\Re^2$ that simplifies enumeration of the regions. That is not the case for bi-parametric CQO problems which results in a more complicated computational algorithm.

In this paper, we consider the bi-parametric optimal partition invariancy sensitivity analysis for CQO in the general case, when both $\Delta b$ and $\Delta c$ are nonzero vectors and parameters $\varepsilon$ and $\lambda$ change independently. We are interested in finding all the regions on the "$\varepsilon-\lambda$" plane where the optimal partition is invariant, i.e., $\pi(\varepsilon,\lambda)=(\mathcal{B}(\varepsilon,\lambda),\mathcal{N}(\varepsilon,\lambda),\mathcal{T}(\varepsilon,\lambda))$. We call each of these regions invariancy region and denote it by $\mathcal{IR}(\Delta b,\Delta c)$. It is obvious that one of these regions includes the origin (0,0), and thus, their union is a nonempty set.

## 2. Fundamental Properties

Here we prove fundamental properties of the invariancy region and describe the behaviour of the optimal value function on this region. First, we prove that this region is a convex set.

**Lemma 1**: The set $\mathcal{IR}(\Delta b,\Delta c)$ is a convex set.

**Proof:** Let $(\varepsilon_1,\lambda_1)$ and $(\varepsilon_2,\lambda_2)$ are two arbitrary pairs in $\mathcal{IR}(\Delta b,\Delta c)$. Let $(x^{(1)},y^{(1)},s^{(1)})$ and $(x^{(2)},y^{(2)},s^{(2)})$ are maximally (strictly) complementary optimal solutions of problems (1) and (2) at these points. Let $(\varepsilon,\lambda)$ be an arbitrary point on the line segment between two points $(\varepsilon_1,\lambda_1)$ and $(\varepsilon_2,\lambda_2)$. Now, we show that the optimal partition at the point $(\varepsilon,\lambda)$ is the same as at $(\varepsilon_1,\lambda_1)$ and $(\varepsilon_2,\lambda_2)$. The details can be found in (Ghaffari-Hadigheh *et al.*, 2008b).

The boundaries between the invariancy regions are line (half-line) segments. The line segment between two adjacent invariancy regions is referred to as *transitional line segment* and the intersection of two transition lines are called *transition points*. Transition points (singleton invariancy regions) and transition lines are called *trivial* invariancy regions. An invariancy region that is neither a singleton nor a transition line is referred to as *non-trivial* invariancy region. Non-trivial invariancy region is a convex polygon that might be unbounded.

**Theorem 2**: The optimal value function is quadratic on the invariancy region $\mathcal{IR}(\Delta b,\Delta c)$.

**Proof:** If the invariancy region is a non-singleton trivial region, then it is a univariate quadratic function by Theorem 4.5 in (Ghaffari-Hadigheh *et al.*, 2007). Consider a non-trivial invariancy region. Utilizing (3) we get the following representation of the optimal value function:

$$\phi(\varepsilon,\lambda)=b_0 + b_1\varepsilon + b_2\lambda + b_3\varepsilon\lambda + b_4\varepsilon^2 + b_5\lambda^2, \tag{4}$$

The formulas for the coefficients in equation (4) can be found in (Ghaffari-Hadigheh *et al.*, 2008b). Clearly (4) is a quadratic function of $\varepsilon$ and $\lambda$ and the claim of the theorem follows directly from (4). The proof is complete.

Recall that the range of parameter variation is an interval of the real line when $\lambda=\varepsilon$. In this case one can identify the range of parameters via solving the following two auxiliary LO problems (Ghaffari-Hadigheh *et al.*, 2007):

$$\lambda_l = \min_{\lambda,x,y,s} \{\lambda : Ax - \lambda\Delta b = b, A^T y + s - Qx - \lambda\Delta c = c, x_{\mathcal{B}} \ge 0, x_{\mathcal{N}\cup\mathcal{T}} = 0, s_{\mathcal{N}} \ge 0, s_{\mathcal{B}\cup\mathcal{T}} = 0\}, \tag{5}$$

and

$$\lambda_u = \max_{\lambda,x,y,s} \{\lambda : Ax - \lambda\Delta b = b, A^T y + s - Qx - \lambda\Delta c = c, x_{\mathcal{B}} \ge 0, x_{\mathcal{N}\cup\mathcal{T}} = 0, s_{\mathcal{N}} \ge 0, s_{\mathcal{B}\cup\mathcal{T}} = 0\}, \tag{6}$$

where $\pi=(\mathcal{B},\mathcal{N},\mathcal{T})$ is the optimal partition for the given initial value of $\varepsilon=\lambda$. Essentially, problems (5) and (6) minimize and maximize $\lambda$ keeping the optimal partition constant. One may find the details in (Ghaffari-Hadigheh *et al.*, 2007).

## 3. Algorithm for Detecting All Invariancy Regions

In this chapter, we describe the tools to identify a non-trivial invariancy region. Recall that for $\varepsilon=\lambda$, the bi-parametric CQO problem reduces to a uni-parametric CQO problem. This trivial observation suggests choosing a method to convert a bi-parametric CQO problem into uni-parametric CQO problems and use Algorithm 1 from (Ghaffari-Hadigheh *et al.*, 2007) to solve those. We start with finding some points on the boundary of the invariancy region. To accomplish this, we select the lines $\varepsilon=t\lambda+h$ and substitute those into problem (1) converting it to the following uni-parametric CQO problem:

$$\min\left\{(c + \lambda\Delta c)^T x + 1/2x^T Qx : Ax = (b + h\Delta b) + \lambda\Delta\underline{b}, x \ge 0\right\} \tag{7}$$

where $\Delta\underline{b}=t\Delta b$. This way we can solve the two associated auxiliary LO problems (5) and (6) to identify the range of variation for parameter $\lambda$ when equation (7) holds.

The algorithm for enumerating all invariancy regions on the "$\varepsilon$–$\lambda$" plane consists of the algorithm for identifying one invariancy region, computing the optimal value function on it and the algorithm for proceeding from one invariancy region to another one, and finally to enumerate all of them. To identify the current invariancy region we start from an inner point of the region (Fig. 1 (a) and (b) illustrates finding an inner point $v_0$) and solve problem (7) a number of times to identify the boundary of the current invariancy region. We are going to "shoot" by solving problem (5) or (6) counter-clockwise from the initial point to identify each transition line (edge), see Fig. 1 (c-f). As we already know one of the edges, we exclude all the angles $\alpha_{\exp}$ between the initial point and the two vertices $v_1$ and $v_2$ of the known edge from the candidate angles to shoot. So, we shoot in the angle $v_0$–$v_2$ plus in the small angles $\beta$ and $2\beta$ and identify the optimal partition in the two points we get. If the optimal partition is the same, we compute the vertices of this new edge $e_2$ and verify that one of those corresponds to the vertex of the previously known edge $e_1$. If it is not the case, then bisection is used to identify the missing edges between $e_1$ and $e_2$. We continue in this manner until all the edges of the invariancy region are identified.

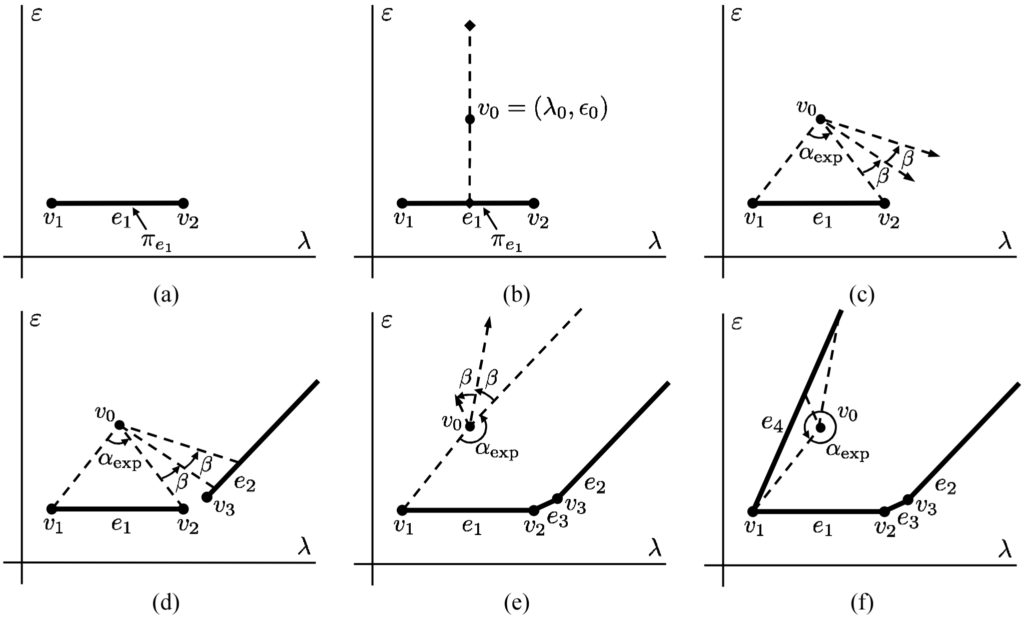


**Fig. 1.** Cell exploration algorithm

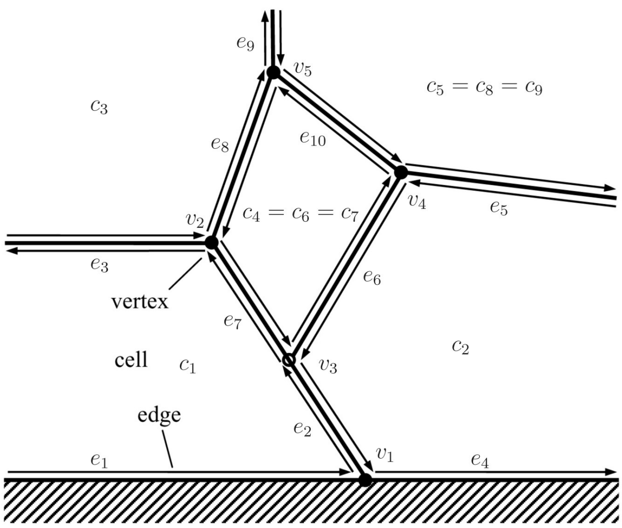


**Fig. 2.** Invariancy region enumeration on the "$\varepsilon$–$\lambda$" plane

To enumerate all invariancy regions we use two queues that store indices of the invariancy regions (cells) that are already investigated and to be processed. At the start of the algorithm, the first cell enters the to-be-processed queue and the queue of completed cells is empty ($c_1$ is entering the to-be-processed queue on Fig. 2). After that, we identify the cell $c_1$ including all faces and vertices starting from the known edge $e_1$ and moving counter-clockwise. Due to the fact that the optimal partition at the edge between the vertices $v_1$ and $v_2$ is the same, we are able to identify only the edge $e_2$ at the moment. Now, when we have identified all edges incident to the cell $c_2$ we can add the potential cells corresponding to each of the edges to the to-be-processed queue, so $e_1 \to \varnothing$ (infeasible), $e_2 \to c_2$, $e_3 \to c_3$. So, we add $c_2$ and $c_3$ to the to-be-processed queue and move $c_1$ to the completed queue. Next, we start processing the cell $c_2$ as the first element of the to-be-processed queue. At this stage, we identify that the edge $e_2$ is shorter than the original one and we split it into two edges – $e_2$ and $e_7$ (note that the edges $e_2$, $e_7$ and vertex $v_3$ have the same optimal partition). As the result of splitting edge $e_2$ into two edges, we need to add the cell $c_4$ that corresponds to the edge $e_7$ to the to-be-processed queue. We get $e_7 \to c_4$, $e_2 \to c_1$ (already processed), $e_4 \to \varnothing$ (infeasible), $e_5 \to c_5$, $e_6 \to c_6$ and add $c_4$, $c_5$, $c_6$ into the to-be-processed queue. Now, $c_2$ is moved to the completed queue. Next in the to-be processed queue is $c_3$ that gives us $e_3 \to c_1$ (already processed), $e_8 \to c_7$ and $e_9 \to c_8$. So, $c_7$ and $c_8$ are added to the to-be-processed queue and $c_3$ is moved to the completed queue. Next, we process $c_4$ and identify that $c_4 = c_6 = c_7$ based on checking known optimal partitions list and identified edges list. Here, $e_{10} \to c_9$. So, $c_4$ is moved to the completed queue and $c_6$, $c_7$ are removed from the to-be-processed queue. We continue until the to-be-processed queue is empty and, consequently, we have identified all the invariancy regions.

The proposed Algorithm 1 runs in linear time in the output size. But, by the nature of the parametric problem, the number of vertices, edges and faces can be exponential in the input size. In our experiences worst case does not happen in practise very often though.

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**Data:** The CQO optimization problem and $\Delta b$, $\Delta c$;
**Result:** Optimal partitions on all invariancy intervals, optimal value function;
**initialization:** compute the problem (1) feasibility bounds in the "$\varepsilon$–$\lambda$" plane and compute inner point in one of the invariancy regions;
**while** *not all invariancy regions are enumerated* **do**
  run sub-algorithm to compute all edges and vertices of the current invariancy region;
  add all unexplored regions corresponding to each edge to the to-be-processed queue and move the current region to the queue of completed region indices;
  **if** *to-be-processed queue of the unexplored regions is not empty* **then**
    pull out the first region from the to-be-processed queue;
    compute an inner point of the new region;
  **else** return the data structure with all the invariancy regions, corresponding optimal partitions and optimal value function;
**end**

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**Algorithm 1.** Algorithm for enumerating all invariancy regions

## 4. Numerical Illustration

Here we present some illustrative numerical results for a simple example by using the algorithm outlined in Chapter 3. Computations can be performed by using any IPM solver for LO and CQO problems due to the fact that IPMs find a maximally complementary solution in the limit. We have used McIPM (Romanko, 2004) and MOSEK for our computations. Let us consider the following CQO problem with $x,c \in \Re^5$, $b \in \Re^3$, $Q \in \Re^{5\times 5}$ being a positive semidefinite symmetric matrix, $A \in \Re^{3\times 5}$ with rank($A$)=3. The problem data is as follows

$$Q=\begin{bmatrix}4&2&0&0&0\\2&5&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\ c=\begin{bmatrix}-16\\-20\\0\\0\\0\end{bmatrix},\ \Delta c=\begin{bmatrix}7\\6\\0\\0\\0\end{bmatrix},\ A=\begin{bmatrix}2&2&1&0&0\\2&1&0&1&0\\2&5&0&0&1\end{bmatrix},\ b=\begin{bmatrix}11\\8\\20\end{bmatrix},\ \Delta b=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

With this data the perturbed CQO instance is

$$\begin{aligned}\min\quad & (-16+7\lambda)x_1+(-20+6\lambda)x_2+2x_1^2+2x_1x_2+5/2x_2^2\\ \text{s.t.}\quad & 2x_1+2x_2+x_3=11+\varepsilon\\ & 2x_1+x_2+x_4=8+\varepsilon\\ & 2x_1+5x_2+x_5=20+\varepsilon\\ & x_1,x_2,x_3,x_4,x_5\ge 0.\end{aligned}\tag{8}$$

The result of our computations is presented in Fig. 3. Fig. 3(a) shows the invariancy regions, the optimal partitions and the equations for the optimal value function. The optimal partitions for the invariancy intervals are shown in ovals, where each letter corresponds to the index being in one of the tri-partition sets $\mathcal{B}$, $\mathcal{N}$ or $\mathcal{T}$. The partitions for the transition points are shown next to them. The solid dots correspond to the cases where the optimal partition in those transition points is different from the partitions on the neighbouring invariancy intervals and invariancy regions. The circle at the point $\lambda$=10/3, $\varepsilon$= –8 corresponds to the case when the optimal partition for the whole line $\varepsilon$= –8 is the same, but it differs for the other segments ending at the point. The graph of $\phi(\lambda,\varepsilon)$ and the corresponding invariancy regions are presented on Fig. 3(b). The piecewise quadratic optimal value function is drawn in different shades that correspond to the invariancy regions.

## 5. Conclusions

We have extended the uni-parametric simultaneous perturbation results for CQO to the bi-parametric case. The algorithm outlined in the paper allows identifying all invariancy regions where the optimal partition is invariant by solving a series of uni-parametric CQO problems. We can also compute the optimal value function and optimal solutions on each invariancy region.

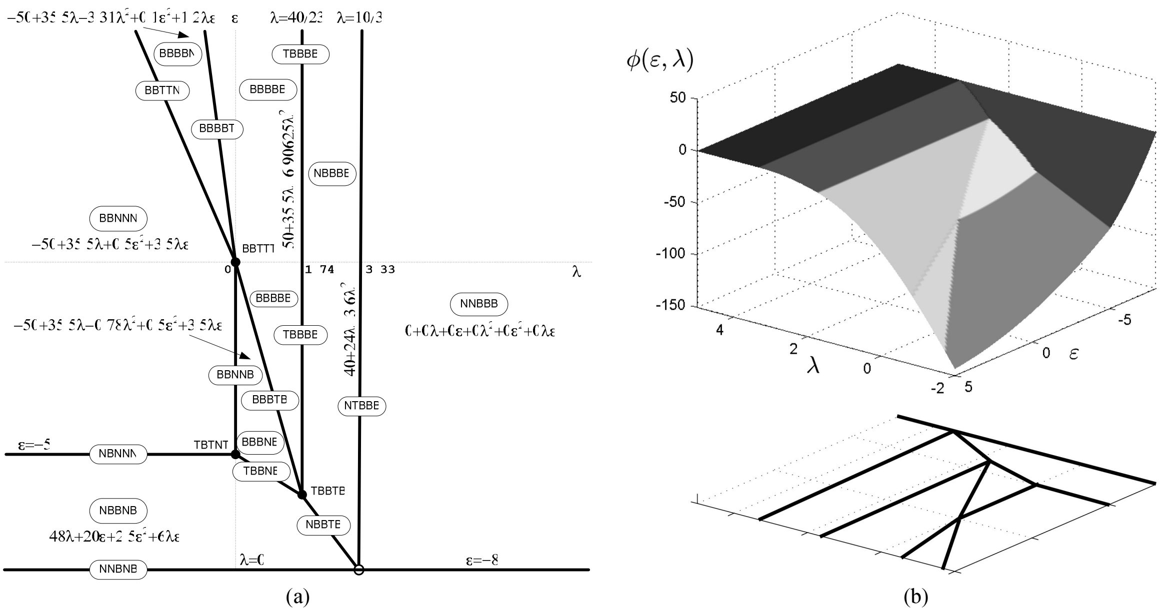


**Fig. 3.** Invariancy regions, optimal partitions and the optimal value function

Even though all presented auxiliary optimization problems can be solved in polynomial time by IPMs and the number of different optimal partitions is finite, enumeration of all invariancy regions may not be achieved in polynomial time due to the fact that the number of different optimal partitions may increase exponentially with the cardinality of the index set. That is why the algorithm presented is linear in the output size, but not in the input size.

Future work inspired by this paper includes extending the methodology of solving bi-parametric programming problems to other classes of optimization problems and improving the implementation. To the best of our knowledge, this paper is the first attempt towards a comprehensive study of bi-parametric convex quadratic optimization problems with different parameters in the coefficients of the objective function and right-hand-side of the constraints.

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